QUESTION BANK

GE 2 , LINEAR ALGEBRA

Q 1 Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$.

(a) Prove that if $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ then $\mathbf{x}\cdot\mathbf{y} = 0$.

(b) Prove that if $\mathbf{x}\cdot\mathbf{y} = 0$ then $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

Q 2 Prove that if $(\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}-\mathbf{y}) = 0$, then $\|\mathbf{x}\| = \|\mathbf{y}\|$.

Q 3 Prove that $\frac{1}{2}\left(\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2\right) = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ for any vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.

Q 4 Prove that $\mathbf{x}\cdot\mathbf{y} = \frac{1}{4}\left(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2\right)$, if $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$.

Q 5 Find the reduced row echelon form $\mathbf{B}$ of the following matrix $\mathbf{A}$, keeping track of the row operations used:

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & -20 \\ -2 & 0 & 11 \\ 3 & 1 & -15 \end{bmatrix}.$$

Q 6 Verify that the following matrices are row equivalent by showing that they have the same reduced row echelon form:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & -3 \\ 0 & -2 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -5 & 3 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

Q 7 Use row reduction to find the inverse, if it exists, for each of the following:

$$\begin{bmatrix} 5 & 7 & -6 \\ 3 & 1 & -2 \\ 1 & -5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 3 \\ 8 & -4 & 9 \\ -4 & 6 & -9 \end{bmatrix}$$

Q 8 Find the characteristic polynomial of

$$\begin{bmatrix} 5 & 1 & 4 \\ 1 & 2 & 3 \\ 3 & -1 & 1 \end{bmatrix}$$

Q 9 Solve for the eigenspace E_λ corresponding to the given eigenvalue λ

$$\begin{bmatrix} -5 & 2 & 0 \\ -8 & 3 & 0 \\ 4 & -2 & -1 \end{bmatrix}, \lambda = -1$$

Q 10 Prove that the set of all scalar multiples of the vector $[1, 3, 2]$ in $\mathbb{R}^3$ forms a vector space with the usual operations on 3-vectors.

Q 11 Prove that $\mathbb{R}$ is a vector space using the operations $\oplus$ and $\odot$ given by $\mathbf{x} \oplus \mathbf{y} = (x^3 + y^3)^{1/3}$ and $a \odot \mathbf{x} = (\sqrt[3]{a})x$.

Q 12 Show that the set $\mathbb{R}^2$, with the usual scalar multiplication but with vector addition replaced by $[x, y] \oplus [w, z] = [x + w, 0]$, does *not* form a vector space.

Q 13 Prove that the set of nonsingular $n \times n$ matrices under the usual operations is *not* a vector space.

Q 14 Show that the set of vectors of the form $[a, b, 0, c, a - 2b + c]$ in $\mathbb{R}^5$ forms a subspace of $\mathbb{R}^5$ under the usual operations.

Q 15 Show that the set of vectors of the form $[2a - 3b, a - 5c, a, 4c - b, c]$ in $\mathbb{R}^5$ forms a subspace of $\mathbb{R}^5$ under the usual operations.

Q 16 Prove that the set of discontinuous real-valued functions defined on $\mathbb{R}$ (for example, $\mathbf{f}(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$) with the usual function operations is not a subspace of the vector space of all real-valued functions with domain $\mathbb{R}$.

Q 17 Prove that the set $S = \{[1,3,-1],[2,7,-3],[4,8,-7]\}$ spans $\mathbb{R}^3$.

Q 18 Prove that the set $S = \{[1,-2,2],[3,-4,-1],[1,-4,9],[0,2,-7]\}$ does not span $\mathbb{R}^3$.

Q 19 Show that the set $\{x^2 + x + 1, x + 1, 1\}$ spans $\mathcal{P}_2$.

Prove that the set $\{x^2 + 4x - 3, 2x^2 + x + 5, 7x - 11\}$ does not span $\mathcal{P}_2$.

Q 20 Show that the following is a linearly dependent subset of $\mathcal{M}_{22}$:

$$\left\{ \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ -6 & 1 \end{bmatrix}, \begin{bmatrix} 4 & -1 \\ -5 & 2 \end{bmatrix}, \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix} \right\}.$$

Q 21 Prove that the following is linearly independent in $\mathcal{M}_{32}$:

$$\left\{ \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ -6 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 7 \\ 5 & 2 \\ -1 & 6 \end{bmatrix} \right\}.$$

Q 22 Prove that the following set is a basis for $\mathcal{M}_{22}$ by showing that it spans $\mathcal{M}_{22}$ and is linearly independent:

$$\left\{ \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 5 & -2 \\ 0 & -3 \end{bmatrix} \right\}.$$

Q 23 Show that the subset $\left\{x^4, x^4 - x^3, x^4 - x^3 + x^2, x^4 - x^3 + x^2 - x, x^3 - 1\right\}$ of $\mathcal{P}_4$ is a basis for $\mathcal{P}_4$.

Q 24 Let $\mathcal{W}$ be the solution set to the matrix equation $\mathbf{AX} = \mathbf{O}$, where

$$\mathbf{A} = \begin{bmatrix} -1 & -2 & -4 & 8 \\ 3 & 4 & 6 & -14 \\ -2 & -1 & 1 & 7 \\ 1 & 0 & -2 & 0 \\ 2 & 3 & 5 & -12 \end{bmatrix}.$$

(a) Show that $\mathcal{W}$ is a subspace of $\mathbb{R}^4$.

(b) Find a basis for $\mathcal{W}$.

(c) Show that $\dim(\mathcal{W}) + \operatorname{rank}(\mathbf{A}) = 4$.